

Dixmier traces are weak* dense in the set of all fully symmetric traces

F. Sukochev

School of Mathematics and Statistics, University of New South Wales, Sydney, 2052, Australia.

D. Zanin*

School of Mathematics and Statistics, University of New South Wales, Sydney, 2052, Australia.

Abstract

We extend Dixmier's construction of singular traces (see [4]) to arbitrary fully symmetric operator ideals. In fact, we show that the set of Dixmier traces is weak* dense in the set of all fully symmetric traces (that is, those traces which respect Hardy-Littlewood submajorization). Our results complement and extend earlier work of Wodzicki [25].

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1. Introduction

In his groundbreaking paper [4], J. Dixmier proved the existence of singular traces (that is, linear positive unitarily invariant functionals which vanish on all finite dimensional operators) on the algebra $B(H)$ of all bounded linear operators acting on infinite-dimensional separable Hilbert space H . Namely, if $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a concave increasing function such that

$$\lim_{t \rightarrow \infty} \frac{\psi(2t)}{\psi(t)} = 1, \quad (1)$$

then there is a singular trace τ_ω , defined for every positive compact operator $A \in B(H)$ by setting

$$\tau_\omega(A) = \omega\left(\frac{1}{\psi(n+1)} \sum_{k=0}^n \mu(k, A)\right). \quad (2)$$

*Corresponding author

Email addresses: f.sukochev@unsw.edu.au (F. Sukochev), d.zanin@unsw.edu.au (D. Zanin)

Here, ω is an arbitrary dilation invariant singular state on the algebra l_∞ of all bounded sequences and $\{\mu(k, A)\}_{k \geq 0}$ is the sequence of singular values of the compact operator $A \in B(H)$ taken in the descending order. This trace is finite on $0 \leq A \in B(H)$ if A belongs to the Marcinkiewicz ideal (see e.g. [9],[10],[19])

$$\mathcal{M}_\psi := \{A \in B(H) : \sup_{n \geq 0} \frac{1}{\psi(n+1)} \sum_{k=0}^n \mu(k, A) < \infty\}.$$

In [13], Dixmier's result and construction was extended to an arbitrary Marcinkiewicz ideal \mathcal{M}_ψ with the following condition on ψ

$$\liminf_{t \rightarrow \infty} \frac{\psi(2t)}{\psi(t)} = 1. \quad (3)$$

All the traces defined above by formula (2) vanish on the ideal \mathcal{L}_1 consisting of all compact operators $A \in B(H)$ such that

$$\text{Tr}(|A|) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \mu(k, A) < \infty.$$

A symmetric operator ideal $\mathcal{E} \subset B(H)$ is a Banach space such that $A \in \mathcal{E}$ and $\{\mu(k, B)\}_{k \geq 0} \leq \{\mu(k, A)\}_{k \geq 0}$ implies that $B \in \mathcal{E}$ and $\|B\|_{\mathcal{E}} \leq \|A\|_{\mathcal{E}}$ (see e.g. [9], [10], [21]¹, [20], [15]).

In analyzing Dixmier's proof of the linearity of τ_ω given by (2), it was observed in [13] (see also [1, 7]) that τ_ω possesses the following fundamental property, namely if $0 \leq A, B \in \mathcal{M}_\psi$ are such that

$$\sum_{k=0}^n \mu(k, B) \leq \sum_{k=0}^n \mu(k, A), \quad \forall n \geq 0, \quad (4)$$

then $\tau_\omega(B) \leq \tau_\omega(A)$. Such a class of traces was termed “fully symmetric” in [15], [22] (see also earlier papers [7],[17], where the term “symmetric” was used). It is natural to consider such traces only on fully symmetric operator ideals \mathcal{E} (that is, on symmetric operator ideals \mathcal{E} satisfying the condition: if A, B satisfy (4) and $A \in \mathcal{E}$, then $B \in \mathcal{E}$ and $\|B\|_{\mathcal{E}} \leq \|A\|_{\mathcal{E}}$). In fact, it was established in [7] that every Marcinkiewicz ideal \mathcal{M}_ψ with ψ satisfying the condition (3) possesses fully symmetric traces.

Furthermore, in the recent paper [13], the following unexpected result was established.

Theorem 1.1. *If ψ satisfies the condition (3), then every fully symmetric trace on \mathcal{M}_ψ is a Dixmier trace τ_ω for some ω .*

¹ We have to caution the reader that in Theorem 1.16 of [21] the assertion (b) does not hold for the norm of an arbitrary symmetric operator ideal \mathcal{E} (see e.g. corresponding counterexamples in [14, p. 83]).

In his seminal paper [25], Wodzicki considered multiplicative renormalisation of positive compact operators. He was probably, the first who suggested that Dixmier construction works on the symmetric operator ideals different from Marcinkiewicz ideals.

More precisely, given a positive function ψ on $(0, \infty)$ (Wodzicki did not assume this function to be either increasing or concave), one can construct a mapping

$$A \rightarrow \left\{ \frac{1}{\psi(n+1)} \sum_{k=0}^n \mu(k, A) \right\}_{n \geq 0}.$$

Applying some limiting procedure to the latter sequence (Wodzicki used Stone-Ćech compactification for this purpose), we are left with a question whether this construction produces a trace. If it does, then it is natural to refer to such a trace as to the Dixmier trace.

Wodzicki proved (a very complicated) criterion for the additivity of multiplicative renormalisation (see Theorem 3.4 in [25]). The questions of finiteness and non-triviality, also considered in [25], happen to be even harder. In fact, it is proved in [25] that, for every principal ideal, multiplicative renormalisation produces a trace if and only if the ideal admits a trace. The latter result still relates to the realm of Marcinkiewicz operator ideals and can be compared with [24, 7, 11, 13].

Our main result extends the above mentioned results of Wodzicki and Theorem 1.1 to an arbitrary fully symmetric operator ideal.

Theorem 1.2. *Let \mathcal{E} be a fully symmetric operator ideal. If \mathcal{E} admits a trace, then there are Dixmier traces on \mathcal{E} . Moreover, those Dixmier traces are weak* dense in the set of all fully symmetric traces on \mathcal{E} .*

The result of Theorem 1.2 is a combination of Theorems 3.4 and 5.2 below.

It should also be pointed out that the result of Theorem 4.6 below substantially strengthens the result of Theorem 3.4 in [25] and is much easier to apply in concrete situations (at least, in the setting of symmetric operator ideals).

2. Preliminaries

The theory of singular traces on symmetric operator ideals rests on some classical analysis which we now review for completeness. For more information, we refer to [23].

Let H be a Hilbert space and let $B(H)$ be the algebra of all bounded operators on H equipped with the uniform norm. For every $A \in B(H)$, one can define a singular value function $\mu(A)$ (see e.g. [8]).

Definition 2.1. *Given an operator $A \in B(H)$, its singular value function $\mu(A)$ is defined by the formula*

$$\mu(t, A) = \inf\{\|Ap\| : \text{Tr}(1 - p) \leq t\}.$$

Clearly, $\mu(A)$ is a step function and, therefore, it can be identified with the sequence of singular numbers of the operators A (the singular values are the eigenvalues of the operator $|A| = (A^*A)^{1/2}$ arranged with multiplicity in decreasing order). That is, we also use the notation $\mu(A) = \{\mu(k, A)\}_{k \geq 0}$.

Equivalently, $\mu(A)$ can be defined in terms of the distribution function d_A of A . That is, setting

$$d_A(s) = \text{Tr}(E^{|A|}(s, \infty)), \quad s \geq 0,$$

we obtain

$$\mu(t, A) = \inf\{s \geq 0 : d_A(s) \leq t\}, \quad t > 0.$$

Here, $E^{|A|}$ denotes the spectral projection of the operator $|A|$.

Further, we need to recall the important notion of Hardy–Littlewood majorization.

Definition 2.2. *The operator $B \in B(H)$ is said to be majorized by the operator $A \in B(H)$ (written $B \prec\prec A$) if and only if*

$$\int_0^t \mu(s, B) ds \leq \int_0^t \mu(s, A) ds, \quad t \geq 0.$$

We have (see [8])

$$A + B \prec\prec \mu(A) + \mu(B) \prec\prec 2\sigma_{1/2}\mu(A + B) \quad (5)$$

for every positive operators $A, B \in B(H)$.

If $s > 0$, the dilation operator $\sigma_s : L_\infty(0, \infty) \rightarrow L_\infty(0, \infty)$ is defined by setting

$$(\sigma_s(x))(t) = x\left(\frac{t}{s}\right), \quad t > 0.$$

Similarly, in the sequence case, we define an operator σ_n by setting

$$\sigma_n(a_0, a_1, \dots) = (\underbrace{a_0, \dots, a_0}_{n \text{ times}}, \underbrace{a_1, \dots, a_1}_{n \text{ times}}, \dots)$$

and an operator $\sigma_{1/2}$ by setting

$$\sigma_{1/2} : (a_0, a_1, a_2, a_3, a_4, \dots) \rightarrow \left(\frac{a_0 + a_1}{2}, \frac{a_2 + a_3}{2}, \dots\right).$$

Below, we define symmetric ideals of l_∞ and that of $B(H)$.

Definition 2.3. *An ideal E of the algebra l_∞ , equipped with the norm $\|\cdot\|_E$, is said to be symmetric if*

1. $(E, \|\cdot\|_E)$ is a Banach space.
2. For every $x \in E$ and every $y \in l_\infty$, we have $\|xy\|_E \leq \|x\|_E \|y\|$.

3. For every $x \in E$ and every permutation $\pi : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+$, we have $x \circ \pi \in E$ and $\|x \circ \pi\|_E = \|x\|_E$.

Definition 2.4. A two-sided ideal \mathcal{E} of the algebra $B(H)$, equipped with the norm $\|\cdot\|_{\mathcal{E}}$, is called *symmetric operator ideal* if

1. $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ is a Banach space.
2. For every $A \in \mathcal{E}$ and every $B \in B(H)$, we have

$$\|AB\|_{\mathcal{E}} \leq \|A\|_{\mathcal{E}}\|B\|, \quad \|BA\|_{\mathcal{E}} \leq \|A\|_{\mathcal{E}}\|B\|.$$

It follows easily from the definition that, for every $A \in \mathcal{E}$ and every $B \in B(H)$ with $\mu(B) \leq \mu(A)$, we have $B \in \mathcal{E}$ and $\|B\|_{\mathcal{E}} \leq \|A\|_{\mathcal{E}}$ (see, e.g. [9]). In particular, we have $\|A\|_{\mathcal{E}} = \|UA\|_{\mathcal{E}} = \|AU\|_{\mathcal{E}}$ for every $A \in \mathcal{E}$ and every unitary $U \in B(H)$. Thus, $\|A\|_{\mathcal{E}} = \|A^*\|_{\mathcal{E}} = \||A|\|_{\mathcal{E}}$ for every $A \in \mathcal{E}$. It is well-known that every proper ideal of the algebra $B(H)$ consists of compact operators.

The following fundamental result appeared in [14].

Theorem 2.5. Let E be a symmetric ideal in l_{∞} . The set

$$\mathcal{E} = \{A \in B(H) : \mu(A) \in E\}$$

equipped with a norm $\|A\|_{\mathcal{E}} = \|\mu(A)\|_E$ is a symmetric operator ideal.

Definition 2.6. Symmetric operator ideal is said to be *fully symmetric* if, for every operator $A \in \mathcal{E}$ and $B \in B(H)$ such that $B \prec\prec A$, we have $B \in \mathcal{E}$ and $\|B\|_{\mathcal{E}} \leq \|A\|_{\mathcal{E}}$.

One should note that every fully symmetric operator ideal is a union of Marcinkiewicz operator ideals (see the text following Theorem II.5.7 of [16]).

Definition 2.7. Let \mathcal{E} be a symmetric operator ideal. A linear functional $\varphi : \mathcal{E} \rightarrow \mathbb{C}$ is said to be a *trace* if $\varphi(U^{-1}AU) = \varphi(A)$ for every $A \in \mathcal{E}$ and every unitary $U \in B(H)$.

One can show that $\varphi(A) = \varphi(B)$ for every positive operators $A, B \in \mathcal{E}$ such that $\mu(A) = \mu(B)$.

Definition 2.8. A trace $\varphi : \mathcal{E} \rightarrow \mathbb{C}$ is called *fully symmetric* if $\varphi(B) \leq \varphi(A)$ for every positive operators $A, B \in \mathcal{E}$ with $B \prec\prec A$.

3. Relatively normal traces are dense

In this section, we introduce an important class of relatively normal traces (see Definition 3.3 below) and prove that they are weak* dense among all fully symmetric traces. The main result of this section is Theorem 3.4.

Let \mathcal{E} be a fully symmetric operator ideal and let φ be a fully symmetric trace on \mathcal{E} . In what follows, \mathcal{E}_+ denotes the positive cone of \mathcal{E} .

Lemma 3.1. *Let $\mathcal{M}_\psi \subset \mathcal{E}$ be a Marcinkiewicz space and let φ be a fully symmetric trace on \mathcal{E} . The mapping $\varphi_{normal,\psi} : \mathcal{E}_+ \rightarrow \mathbb{R}$ defined by setting*

$$\varphi_{normal,\psi}(A) = \sup\{\varphi(B) : B \in \mathcal{M}_\psi, 0 \leq B \prec\prec A\}, \quad 0 \leq A \in \mathcal{E}, \quad (6)$$

is additive on the positive cone of \mathcal{E} .

Proof. Let $A_1, A_2 \in \mathcal{E}_+$. Let $B \in \mathcal{M}_\psi$ be such that $0 \leq B \prec\prec A_1 + A_2$. By [5, Theorem 2.2], there exists a linear operator $\mathbf{C} : B(H) \rightarrow B(H)$ (a positive contraction both in $B(H)$ and in \mathcal{L}_1) such that $B = \mathbf{C}(A_1 + A_2)$. Setting $B_1 = \mathbf{C}(A_1) \geq 0$ and $B_2 = \mathbf{C}(A_2) \geq 0$, we have $B = B_1 + B_2$. Therefore, $0 \leq B_i \leq B \in \mathcal{M}_\psi$ and $B_i \prec\prec A_i$. Hence, by definition (6),

$$\varphi(B) = \varphi(B_1) + \varphi(B_2) \leq \varphi_{normal,\psi}(A_1) + \varphi_{normal,\psi}(A_2).$$

Taking the supremum over all B in question, we obtain

$$\varphi_{normal,\psi}(A_1 + A_2) \leq \varphi_{normal,\psi}(A_1) + \varphi_{normal,\psi}(A_2). \quad (7)$$

Fix $\varepsilon > 0$. There exist $B_i \in \mathcal{M}_\psi$ such that $0 \leq B_i \prec\prec A_i$ and $\varphi(B_i) > \varphi_{normal,\psi}(A_i) - \varepsilon$. In particular, we have

$$\varphi_{normal,\psi}(A_1) + \varphi_{normal,\psi}(A_2) \leq 2\varepsilon + \varphi(B_1 + B_2). \quad (8)$$

Further, we have

$$B_1 + B_2 \prec\prec \mu(B_1) + \mu(B_2) \prec\prec \mu(A_1) + \mu(A_2) \prec\prec 2\sigma_{1/2}\mu(A_1 + A_2).$$

It follows from (8) and definition (6) that

$$\begin{aligned} & \varphi_{normal,\psi}(A_1) + \varphi_{normal,\psi}(A_2) \leq \\ & \leq 2\varepsilon + \varphi_{normal,\psi}(2\sigma_{1/2}\mu(A_1 + A_2)) = 2\varepsilon + \varphi_{normal,\psi}(A_1 + A_2). \end{aligned}$$

Here, the last equality follows from Lemma 4.5 below. Since ε is arbitrarily small, we have

$$\varphi_{normal,\psi}(A_1) + \varphi_{normal,\psi}(A_2) \leq \varphi_{normal,\psi}(A_1 + A_2). \quad (9)$$

The assertion follows from (7) and (9). \square

It is proved in the following lemma that $\varphi_{normal,\psi}$ can be viewed as a “normal part” of the trace φ with respect to the subspace \mathcal{M}_ψ .

Lemma 3.2. *The mapping $\varphi_{normal,\psi} : \mathcal{E}_+ \rightarrow \mathbb{R}$ extends to a fully symmetric trace on \mathcal{E} . Moreover, $\varphi = \varphi_{normal,\psi}$ on \mathcal{M}_ψ and $(\varphi_{normal,\psi})_{normal,\psi} = \varphi_{normal,\psi}$ on \mathcal{E} .*

Proof. Every additive functional on \mathcal{E}_+ uniquely extends to a linear functional on \mathcal{E} . In particular, so does $\varphi_{normal,\psi} : \mathcal{E}_+ \rightarrow \mathbb{R}$.

Let $A_1, A_2 \in \mathcal{E}$ be positive operators such that $A_2 \prec\prec A_1$. It follows that

$$\{\varphi(B) : B \in \mathcal{M}_\psi, 0 \leq B \prec\prec A_2\} \subset \{\varphi(B) : B \in \mathcal{M}_\psi, 0 \leq B \prec\prec A_1\}.$$

Therefore, $\varphi_{normal,\psi}(A_2) \leq \varphi_{normal,\psi}(A_1)$. Hence, $\varphi_{normal,\psi}$ is a fully symmetric trace on \mathcal{E} .

The second assertion is obvious. In order to prove the third assertion, fix a positive operator $A \in \mathcal{E}$. By definition, $(\varphi_{normal,\psi})_{normal,\psi}(A) \leq \varphi_{normal,\psi}(A)$. Select $B_m \in \mathcal{M}_\psi$ such that $0 \leq B_m \prec\prec A$ and such that $\varphi(B_m) \rightarrow \varphi_{normal,\psi}(A)$. Clearly, $\varphi_{normal,\psi}(B_m) = \varphi(B_m)$. Thus, $\varphi_{normal,\psi}(B_m) \rightarrow \varphi_{normal,\psi}(A)$. Therefore, $(\varphi_{normal,\psi})_{normal,\psi}(A) \geq \varphi_{normal,\psi}(A)$, and the third assertion is proved. \square

Definition 3.3. A fully symmetric trace φ on \mathcal{E} is called *relatively normal* if there exists a Marcinkiewicz space $\mathcal{M}_\psi \subset \mathcal{E}$ such that $\varphi = \varphi_{normal,\psi}$.

Theorem 3.4. Relatively normal traces on \mathcal{E} are weak* dense in the set of all fully symmetric traces on \mathcal{E} .

Proof. The set $E_+^\downarrow = \{\mu(A), A \in \mathcal{E}\}$, equipped with the partial ordering given by the Hardy-Littlewood majorization is a directed set. For every $x \in E_+^\downarrow$, let $\psi_x : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a concave increasing function such that $\psi'_x = x$. For every given fully symmetric functional φ , consider the net $\{\varphi_{n,\psi_x} \in \mathcal{E}^*, x \in E_+^\downarrow\}$. We claim that this net weak* converges to the functional φ .

Recall that the base of weak* topology (that is, $\sigma(\mathcal{E}^*, \mathcal{E})$) is formed by the sets

$$N(A_1, \dots, A_m, \varepsilon) = \{\theta \in \mathcal{E}^* : |\theta(A_k)| < \varepsilon, 1 \leq k \leq m\}.$$

Fix some neighborhood U of 0 in the weak* topology. Select $\varepsilon > 0$ and operators $0 \leq A_k \in \mathcal{E}$ such that

$$N(A_1, \dots, A_m, \varepsilon) \subset U.$$

Set $y = \sum_{k=1}^m \mu(A_k)$. It is clear that, for every $x \in E_+^\downarrow$ such that $y \prec\prec x$, we have $A_k \in \mathcal{M}_{\psi_x}$. It follows from Lemma 3.2 that $(\varphi_{n,\psi_x} - \varphi)(A_k) = 0$. Therefore,

$$\varphi - \varphi_{n,\psi_x} \in \{\theta \in \mathcal{E}^* : |\theta(A_k)| = 0, 1 \leq k \leq m\} \subset U.$$

\square

4. Dixmier traces

In this section, we introduce the concept of Dixmier trace on symmetric operator ideals. The main result of this section is Theorem 4.6.

4.1. Extension of states

As usual, a state on the algebra l_∞ is a positive linear functional ω such that $\omega(1) = 1$. A state ω is called singular if it vanishes on all finitely supported sequences. A state ω is called dilation invariant if $\omega \circ \sigma_n = \omega$, $n \in \mathbb{N}$.

Lemma 4.1. *Every state ω on the algebra l_∞ admits an extension to an additive mapping ω from the set \mathfrak{L}_+ of the positive (unbounded) sequences to $\mathbb{R}_+ \cup \{\infty\}$. This extension is defined by setting*

$$\omega(x) = \sup\{\omega(y), 0 \leq y \leq x, y \in l_\infty\}, \quad 0 \leq x \in \mathfrak{L}_+. \quad (10)$$

Proof. Let $x_1, x_2 \in \mathfrak{L}_+$. If $0 \leq y \in l_\infty$ is such that $y \leq x_1 + x_2$, then there exist positive elements $y_1, y_2 \in l_\infty$ such that $y = y_1 + y_2$, $y_1 \leq x_1$ and $y_2 \leq x_2$. It follows from (10) that

$$\omega(y) = \omega(y_1) + \omega(y_2) \leq \omega(x_1) + \omega(x_2).$$

Taking the supremum over all such y , we obtain

$$\omega(x_1 + x_2) \leq \omega(x_1) + \omega(x_2). \quad (11)$$

Now, we prove the converse inequality. The latter becomes trivial if $\omega(x_1) = \infty$ or $\omega(x_2) = \infty$. Thus, we may assume without loss of generality that both $\omega(x_1) < \infty$ and $\omega(x_2) < \infty$. Fix $\varepsilon > 0$. Let $y_i \in l_\infty$ be such that $0 \leq y_i \leq x_i$ and $\omega(y_i) > \omega(x_i) - \varepsilon$. It follows from (10) that

$$\omega(x_1) + \omega(x_2) \leq 2\varepsilon + \omega(y_1) + \omega(y_2) = 2\varepsilon + \omega(y_1 + y_2) \leq 2\varepsilon + \omega(x_1 + x_2).$$

Since ε is arbitrarily small, we obtain

$$\omega(x_1) + \omega(x_2) \leq \omega(x_1 + x_2). \quad (12)$$

The assertion follows from (11) and (12). \square

It follows directly from the definition (10) that the extension $\omega : \mathfrak{L}_+ \rightarrow \mathbb{R} \cup \{\infty\}$ defined in Lemma 4.1 is dilation invariant if and only if $\omega : l_\infty \rightarrow \mathbb{R}$ is dilation invariant.

Lemma 4.2. *For every state ω on the algebra l_∞ and for every $x \in \mathfrak{L}_+$, we have*

$$\omega(\min\{n, x\}) \rightarrow \omega(x)$$

as $n \rightarrow \infty$.

Proof. Fix a sequence $\{x_n\}_{n \geq 0} \subset l_\infty$ such that $x_n \leq x$ for every $n \geq 0$, and such that $\omega(x_n) \rightarrow \omega(x)$ as $n \rightarrow \infty$. Evidently, $x_n \leq \min\{\|x_n\|_\infty, x\} \leq x$. It follows that $\omega(\min\{\|x_n\|_\infty, x\}) \rightarrow \omega(x)$ as $n \rightarrow \infty$. If $\|x_n\|_\infty \rightarrow \infty$, then we conclude the proof. If $\|x_n\|_\infty \leq C$ for $n \geq 0$, then $\omega(x) = \omega(\min\{n, x\})$ for every $n \geq C$ and the assertion follows. \square

Lemma 4.3. *Let ω be a state on the algebra l_∞ . Let $0 \leq z \in l_\infty$ be such that $\omega(z) = 0$ and let $u \in \mathfrak{L}_+$ be such that $\omega(u) < \infty$. It follows that $\omega(uz) = 0$.*

Proof. It is clear that $u = \min\{n, u\} + (u - n)_+$. It follows from Lemma 4.2 that the sequence $\omega(\min\{n, u\})$ converges to $\omega(u)$. Since $\omega(u) < \infty$, it follows that $\omega((u - n)_+) \rightarrow 0$. On the other hand, we have $uz \leq nz + \|z\|_\infty(u - n)_+$. Since $\omega(z) = 0$, it follows that $\omega(uz) \leq \|z\|_\infty \omega((u - n)_+)$. Passing $n \rightarrow \infty$, we conclude the proof. \square

4.2. Dixmier traces

Definition 4.4. *Let \mathcal{E} be a fully symmetric operator ideal. For a given concave increasing function ψ with $\mathcal{M}_\psi \subset \mathcal{E}$ and given dilation invariant singular state ω on l_∞ define a mapping $\tau_\omega : \mathcal{E}_+ \rightarrow \mathbb{R}_+ \cup \{\infty\}$ by setting*

$$\tau_\omega(A) = \omega\left(\frac{1}{\psi(n+1)} \sum_{k=0}^n \mu(k, A)\right), \quad 0 \leq A \in \mathcal{E},$$

where the extension of ω to \mathcal{L}_+ is given by Lemma 4.1. If the mapping τ_ω is finite and additive on \mathcal{E}_+ , then its linear extension to \mathcal{E} is called a Dixmier trace on \mathcal{E} .

Lemma 4.5. *Let \mathcal{E} be a symmetric operator ideal and let φ be a trace on \mathcal{E} . For every positive $A \in \mathcal{E}$, we have $\varphi(2\sigma_{1/2}\mu(A)) = \varphi(A)$.*

Proof. Without loss of generality, we can take $A = \mu(A)$. Thus,

$$A = \text{diag}(\mu(0, A), 0, \mu(2, A), 0, \dots) + \text{diag}(0, \mu(1, A), 0, \mu(3, A), \dots).$$

Similarly,

$$2\sigma_{1/2}\mu(A) = \text{diag}(\mu(0, A), \mu(2, A), \dots) + \text{diag}(\mu(1, A), \mu(3, A), \dots).$$

The assertion follows immediately. \square

The following theorem gives a necessary and sufficient condition for the mapping τ_ω to be a trace on \mathcal{E} .

Theorem 4.6. *Let \mathcal{E} be a symmetric operator ideal and let $\mathcal{M}_\psi \subset \mathcal{E}$. Let ω be a singular state on the algebra l_∞ such that τ_ω is finite on \mathcal{E}_+ . The mapping τ_ω is additive on \mathcal{E}_+ if and only if*

$$\omega\left(\frac{\psi(2n+1)}{\psi(n+1)}\right) = 1. \quad (13)$$

Proof. Note that concave function ψ is subadditive and, therefore,

$$\left\{\frac{\psi(2n+1)}{\psi(n+1)}\right\}_{n \geq 0} \in l_\infty.$$

Suppose first that τ_ω is additive on \mathcal{E}_+ . Set

$$A = \text{diag}(\psi(1), \psi(2) - \psi(1), \psi(3) - \psi(2), \dots).$$

Note that $A \in \mathcal{M}_\psi \subset \mathcal{E}$. It is obvious from the definition of A that

$$\omega\left(\frac{\psi(2n+1)}{\psi(n+1)}\right) = \tau_\omega(2\sigma_{1/2}\mu(A)).$$

The equality (13) follows now from Lemma 4.5.

Assume now that the equality (13) holds. Let $A, B \in \mathcal{E}$ be positive operators. It follows from the left hand side inequality in (5) that

$$\tau_\omega(A+B) \leq \tau_\omega(A) + \tau_\omega(B). \quad (14)$$

In order to prove converse inequality, introduce the positive sequence

$$z = \left\{ 1 - \frac{\psi(n+1)}{\psi(2n+1)} \right\}_{n \geq 0}.$$

By the assumption, we have $\omega(z) = 0$. By Definition 4.4 and Lemma 4.3, we have

$$\begin{aligned} \tau_\omega(A) + \tau_\omega(B) &= \omega\left(\frac{1}{\psi(n+1)} \sum_{k=0}^n \mu(k, A) + \mu(k, B)\right) = \\ &= \omega\left((1-z_n) \frac{1}{\psi(n+1)} \sum_{k=0}^n \mu(k, A) + \mu(k, B)\right) = \omega\left(\frac{1}{\psi(2n+1)} \sum_{k=0}^n \mu(k, A) + \mu(k, B)\right). \end{aligned}$$

Applying now the right hand side inequality in (5), we obtain

$$\tau_\omega(A) + \tau_\omega(B) \leq \omega\left(\frac{1}{\psi(2n+1)} \sum_{k=0}^{2n+1} \mu(k, A+B)\right).$$

Since ω is dilation invariant, it follows that

$$\tau_\omega(A) + \tau_\omega(B) \leq (\omega \circ \sigma_2)\left(\frac{1}{\psi(2n+1)} \sum_{k=0}^{2n+1} \mu(k, A+B)\right).$$

However,

$$\sigma_2 \left\{ \frac{1}{\psi(2n+1)} \sum_{k=0}^{2n+1} \mu(k, A+B) \right\}_{n \geq 0} \in \left\{ \frac{1}{\psi(n+1)} \sum_{k=0}^n \mu(k, A+B) \right\}_{n \geq 0} + c_0.$$

Since $\omega|_{c_0} = 0$, it follows that

$$\tau_\omega(A) + \tau_\omega(B) \leq \tau_\omega(A+B). \quad (15)$$

The assertion follows now from (14) and (15). \square

5. Relatively normal traces are Dixmier traces

In this section we prove that every relatively normal trace on symmetric operator ideal must be the Dixmier trace. The main result of this section is Theorem 5.2.

Let $A, B \in B(H)$ be positive operators. Let $A \wedge B$ be any positive operator from $B(H)$ such that

$$\sum_{k=0}^n \mu(k, A \wedge B) = \min\left\{\sum_{k=0}^n \mu(k, A), \sum_{k=0}^n \mu(k, B)\right\}, \quad n \geq 0. \quad (16)$$

Lemma 5.1. *Let \mathcal{E} be a fully symmetric operator ideal and let φ be a relatively normal fully symmetric trace on \mathcal{E} . There exists a positive operator $B \in \mathcal{E}$ such that*

$$\varphi(A) = \lim_{n \rightarrow \infty} \varphi(A \wedge nB), \quad 0 \leq A \in \mathcal{E}.$$

Proof. By assumption and Definition 3.3, there exists a Marcinkiewicz subspace $\mathcal{M}_\psi \subset \mathcal{E}$ such that $\varphi = \varphi_{normal, \psi}$. Set

$$B = \text{diag}(\psi(1), \psi(2) - \psi(1), \psi(3) - \psi(2), \dots).$$

Obviously, $B \in \mathcal{M}_\psi \subset \mathcal{E}$. For every positive $A \in \mathcal{E}$, we have

$$\begin{aligned} \varphi(A) &= \varphi_{normal, \psi}(A) = \sup\{\varphi(C) : C \in \mathcal{M}_\psi, 0 \leq C \prec\prec A\} = \\ &= \lim_{n \rightarrow \infty} \sup\{\varphi(C) : \|C\|_{\mathcal{M}_\psi} \leq n, 0 \leq C \prec\prec A\}. \end{aligned}$$

It follows now from the definition of Marcinkiewicz operator ideal that

$$\begin{aligned} \varphi(A) &= \lim_{n \rightarrow \infty} \sup\{\varphi(C) : C \prec\prec nB, 0 \leq C \prec\prec A\} = \\ &= \lim_{n \rightarrow \infty} \sup\{\varphi(C) : 0 \leq C \prec\prec A \wedge nB\}. \end{aligned}$$

Since the trace φ is fully symmetric, it follows that

$$\varphi(A) = \lim_{n \rightarrow \infty} \varphi(A \wedge nB).$$

□

Theorem 5.2. *Let \mathcal{E} be a fully symmetric operator ideal and let φ be a relatively normal (with respect to the Marcinkiewicz space $\mathcal{M}_\psi \subset \mathcal{E}$) fully symmetric trace on \mathcal{E} . There exists a dilation invariant singular state ω on l_∞ (extended to \mathcal{L}_+ by Lemma 4.1) such that*

$$\varphi(A) = \omega\left(\frac{1}{\psi(n+1)} \sum_{k=0}^n \mu(k, A)\right), \quad 0 \leq A \in \mathcal{E}.$$

Proof. The functional $\varphi|_{\mathcal{M}_\psi}$ is fully symmetric. By Theorem 1.1, $\varphi|_{\mathcal{M}_\psi}$ is a Dixmier trace. In particular, there exists a dilation invariant singular state ω on l_∞ such that

$$\varphi(C) = \omega\left(\frac{1}{\psi(n+1)} \sum_{k=0}^n \mu(k, C)\right), \quad 0 \leq C \in \mathcal{M}_\psi. \quad (17)$$

For every positive $A \in \mathcal{E}$, define $\mathbf{T}(A) \in \mathfrak{L}_+$ by setting

$$\mathbf{T}(A) = \left\{ \frac{1}{\psi(n+1)} \sum_{k=0}^n \mu(k, A) \right\}_{n \geq 0}.$$

Set

$$B = \text{diag}(\psi(1), \psi(2) - \psi(1), \psi(3) - \psi(2), \dots).$$

Obviously, $B \in \mathcal{M}_\psi \subset \mathcal{E}$. Dividing the equality (16) by $\psi(n+1)$ and taking into account that

$$\frac{1}{\psi(n+1)} \sum_{k=0}^n \mu(k, B) = 1,$$

we obtain $\mathbf{T}(A \wedge nB) = \min\{\mathbf{T}(A), n\}$. It follows from (17) and Lemma 5.1 that

$$\varphi(A) = \lim_{n \rightarrow \infty} \omega(\min\{\mathbf{T}(A), n\}), \quad 0 \leq A \in \mathcal{E}.$$

By Lemma 4.2, we conclude that $\varphi(A) = \omega(\mathbf{T}(A))$ for every $0 \leq A \in \mathcal{E}$. \square

6. Wodzicki representation of Dixmier traces

In this section, we prove that every relatively normal trace on a symmetric operator ideal can be represented in the form proposed by Wodzicki [25]. The main result of this section is Theorem 6.3.

The Banach space l_∞ is a commutative C^* -algebra. Let $\beta\mathbb{N}$ be the set of all nontrivial homomorphic functionals on l_∞ . Clearly, $\beta\mathbb{N}$ is a weak* closed subset of a unit ball of l_∞^* . By the Banach-Alaoglu theorem, the unit ball of l_∞^* (and, therefore, the set $\beta\mathbb{N}$) is weak* compact. By Gelfand-Naimark theorem, l_∞ is isometrically isomorphic (via Gelfand transform) to the C^* -algebra of all continuous functions on $\beta\mathbb{N}$. The set $\beta\mathbb{N}$ is usually called the Stone-Ćech compactification of \mathbb{N} . The set $\mathbb{N}_\infty = \beta\mathbb{N} \setminus \mathbb{N}$ is frequently referred to as to the set of all infinite integers.

Lemma 6.1. *Dilation semigroup σ_n , $n \geq 1$, acts on \mathbb{N}_∞ . Every dilation invariant singular state admits a representation*

$$\omega(x) = \int_{\mathbb{N}_\infty} x(p) d\nu(p), \quad x \in l_\infty.$$

with ν being a finite regular dilation invariant Borel measure ν on \mathbb{N}_∞ .

Proof. Let $e_k = \{\delta_{kj}\}_{j \geq 0}$, $k \geq 0$, be the standard basic sequence in l_∞ . If $p \in \beta\mathbb{N}$, then

$$p(e_k)p(e_l) = p(e_k e_l) = 0, \quad k \neq l.$$

In particular, at most one of the numbers $p(e_k)$, $k \geq 0$, is nonzero. Obviously, $p \in \mathbb{N}_\infty$ if and only if $p(e_k) = 0$ for all $k \geq 0$.

For every $p \in \beta\mathbb{N}$ and every $n \in \mathbb{N}$, the mapping $x \rightarrow (\sigma_n x)(p)$ is a homomorphism. Hence, it corresponds to a point $q \in \beta\mathbb{N}$. If $q \notin \mathbb{N}_\infty$, then there exists $k \geq 0$ such that $q(e_k) \neq 0$. Thus,

$$\sum_{m=kn}^{(k+1)n-1} p(e_k) = p\left(\sum_{m=kn}^{(k+1)n-1} e_k\right) = p(\sigma_n e_k) = q(e_k) \neq 0.$$

Hence, $p(e_m) \neq 0$ for some $kn \leq m < (k+1)n$. Thus, $p \notin \mathbb{N}_\infty$. It follows that σ_n acts on \mathbb{N}_∞ .

By Riesz-Markov theorem (see [18]), for every state ω on l_∞ , there exists a finite regular Borel measure ν on $\beta\mathbb{N}$ such that

$$\omega(x) = \int_{\beta\mathbb{N}} x(p) d\nu(p). \quad (18)$$

Kakutani and Nakamura noted in [12] that if the state ω is singular, then the measure ν is supported on \mathbb{N}_∞ .

For every $x \in l_\infty$, we have

$$(\omega \circ \sigma_n)(x) = \int_{\mathbb{N}_\infty} (\sigma_n x)(p) d\nu(p) = \int_{\mathbb{N}_\infty} x(p) d(\nu \circ (\sigma_n)^{-1})(p). \quad (19)$$

Since $\omega = \omega \circ \sigma_n$ for all $n \geq 1$, it follows from (18) and (19) that the measure ν is invariant with respect to the action of the dilation semigroup. \square

Corollary 6.2. *Let ω be a dilation invariant singular state. There exists a finite regular dilation invariant Borel measure ν on \mathbb{N}_∞ such that*

$$\omega(x) = \int_{\mathbb{N}_\infty} x(p) d\nu(p)$$

for every $x \in \mathfrak{L}_+$.

Proof. Fix $p \in \mathbb{N}_\infty$. Extend p to an additive functional on \mathfrak{L}_+ by Lemma 4.1. For every $x \in \mathfrak{L}_+$ and for every $n \in \mathbb{N}$, we have $(\min\{x, n\})(p) = \min\{x(p), n\}$.

For a given $n \in \mathbb{N}$, it follows from above and from Lemma 6.1 that

$$\omega(\min\{x, n\}) = \int_{\mathbb{N}_\infty} \min\{x(p), n\} d\nu(p).$$

It follows from Levi theorem that

$$\int_{\mathbb{N}_\infty} \min\{x(p), n\} d\nu(p) \rightarrow \int_{\mathbb{N}_\infty} x(p) d\nu(p).$$

The assertion follows now from Lemma 4.1. \square

Theorem 6.3. *Let \mathcal{E} be a fully symmetric operator ideal and let φ be a relatively normal (with respect to the Marcinkiewicz space $\mathcal{M}_\psi \subset \mathcal{E}$) fully symmetric trace on \mathcal{E} . There exists a finite regular dilation invariant Borel measure ν on \mathbb{N}_∞ such that*

$$\varphi(A) = \int_{\mathbb{N}_\infty} \left(\frac{1}{\psi(n+1)} \sum_{k=0}^n \mu(k, A) \right) (p) d\nu(p), \quad 0 \leq A \in \mathcal{E}. \quad (20)$$

Proof. The assertion follows immediately from Theorem 5.2 and Corollary 6.2. \square

The following assertion is an easy corollary of Theorem 6.3.

Corollary 6.4. *Let \mathcal{E} be a fully symmetric operator ideal. One of the following mutually exclusive possibilities holds.*

1. *For every $A \in \mathcal{E}$, we have*

$$\frac{1}{n} \|A^{\oplus n}\|_{\mathcal{E}} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

2. *There exist a concave increasing function ψ and a finite regular dilation invariant Borel measure ν on \mathbb{N}_∞ such that the mapping*

$$A \rightarrow \int_{\mathbb{N}_\infty} \left(\frac{1}{\psi(n+1)} \sum_{k=0}^n \mu(k, A) \right) (p) d\nu(p), \quad 0 \leq A \in \mathcal{E}$$

extends to a trace on \mathcal{E} .

Proof. Suppose that there exists $A \in \mathcal{E}$ such that

$$\frac{1}{n} \|A^{\oplus n}\|_{\mathcal{E}} \not\rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

It follows from Theorem 5 (ii) in [23] that there exists a fully symmetric trace on \mathcal{E} . By Theorem 3.4, it can be approximated by a relatively normal fully symmetric trace on \mathcal{E} . The assertion follows immediately from Theorem 6.3. \square

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